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# An inverse scattering problem for short-range systems in a time-periodic electric field.

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## Abstract

We consider the time-dependent Hamiltonian  $H(t) = \frac{1}{2}p^2 - E(t) \cdot x + V(t, x)$  on  $L^2(\mathbb{R}^n)$ , where the external electric field  $E(t)$  and the short-range electric potential  $V(t, x)$  are time-periodic with the same period. It is well-known that the short-range notion depends on the mean value  $E_0$  of the external electric field. When  $E_0 = 0$ , we show that the high energy limit of the scattering operators determines uniquely  $V(t, x)$ . When  $E_0 \neq 0$ , the same result holds in dimension  $n \geq 3$  for generic short-range potentials. In dimension  $n = 2$ , one has to assume a stronger decay on the electric potential.

## 1 Introduction.

In this note, we study an inverse scattering problem for a two-body short-range system in the presence of an external time-periodic electric field  $E(t)$  and a time-periodic short-range potential  $V(t, x)$  (with the same period  $T$ ). For the sake of simplicity, we assume that the period  $T = 1$ .

The corresponding Hamiltonian is given on  $L^2(\mathbb{R}^n)$  by :

$$(1.1) \quad H(t) = \frac{1}{2}p^2 - E(t) \cdot x + V(t, x),$$

where  $p = -i\partial_x$ . When  $E(t) = 0$ , the Hamiltonian  $H(t)$  describes the dynamics of the hydrogen atom placed in a linearly polarized monochromatic electric field, or a light particle in the restricted three-body problem in which two other heavy particles are set on prescribed periodic orbits. When  $E(t) = \cos(2\pi t) E$  with  $E \in \mathbb{R}^n$ , the Hamiltonian describes the well-known AC-Stark effect in the  $E$ -direction [7].

In this paper, we assume that the external electric field  $E(t)$  satisfies :

$$(A_1) \quad t \rightarrow E(t) \in L^1_{loc}(\mathbb{R}; \mathbb{R}^n) \quad , \quad E(t+1) = E(t) \text{ a.e.}$$

Moreover, we assume that the potential  $V \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ , is time-periodic with period 1, and satisfies the following estimations :

$$(A_2) \quad \forall \alpha \in \mathbb{N}^n, \forall k \in \mathbb{N}, \quad |\partial_t^k \partial_x^\alpha V(t, x)| \leq C_{k,\alpha} \langle x \rangle^{-\delta-|\alpha|}, \text{ with } \delta > 0,$$

where  $\langle x \rangle = (1 + x^2)^{\frac{1}{2}}$ . Actually, we can accommodate more singular potentials (see [10], [11], [12] for example) and we need  $(A_2)$  for only  $k, \alpha$  with finite order. It is well-known that under assumptions  $(A_1) - (A_2)$ ,  $H(t)$  is essentially self-adjoint on  $\mathcal{S}(\mathbb{R}^n)$  the Schwartz space, [16]. We denote  $H(t)$  the self-adjoint realization with domain  $D(H(t))$ .

Now, let us recall some well-known results in scattering theory for time-periodic electric fields. We denote  $H_0(t)$  the free Hamiltonian :

$$(1.2) \quad H_0(t) = \frac{1}{2}p^2 - E(t) \cdot x \quad ,$$

and let  $U_0(t, s)$ , (resp.  $U(t, s)$ ) be the unitary propagator associated with  $H_0(t)$ , (resp.  $H(t)$ ) (see section 2 for details).

For short-range potentials, the wave operators are defined for  $s \in \mathbb{R}$  and  $\Phi \in L^2(\mathbb{R}^n)$  by :

$$(1.3) \quad W^\pm(s) \Phi = \lim_{t \rightarrow \pm\infty} U(s, t) U_0(t, s) \Phi.$$

We emphasize that the short-range condition depends on the value of the mean of the external electric field :

$$(1.4) \quad E_0 = \int_0^1 E(t) dt \quad .$$

• **The case  $E_0 = 0$ .**

By virtue of the Avron-Herbst formula (see section 2), this case falls under the category of two-body systems with time-periodic potentials and this case was studied by Kitada and Yajima ([10], [11]), Yokoyama [22].

We recall that for a unitary or self-adjoint operator  $U$ ,  $\mathcal{H}_c(U)$ ,  $\mathcal{H}_{ac}(U)$ ,  $\mathcal{H}_{sc}(U)$  and  $\mathcal{H}_p(U)$  are, respectively, continuous, absolutely continuous, singular continuous and point spectral subspace of  $U$ .

We have the following result ([10], [11], [21]) :

**Theorem 1**

Assume that hypotheses  $(A_1)$ ,  $(A_2)$  are satisfied with  $\delta > 1$  and with  $E_0 = 0$ .

Then : (i) the wave operators  $W^\pm(s)$  exist for all  $s \in \mathbb{R}$ .

(ii)  $W^\pm(s+1) = W^\pm(s)$  and  $U(s+1, s) W^\pm(s) = W^\pm(s) U_0(s+1, s)$ .

(iii)  $\text{Ran } (W^\pm(s)) = \mathcal{H}_{ac}(U(s+1, s))$  and  $\mathcal{H}_{sc}(U(s+1, s)) = \emptyset$ .

(iv) the purely point spectrum  $\sigma_p(U(s+1, s))$  is discrete outside  $\{1\}$ .

**Comments.**

1 - The unitary operators  $U(s+1, s)$  are called the Floquet operators and they are mutually equivalent. The Floquet operators play a central role in the analysis of time periodic systems. The eigenvalues of these operators are called Floquet multipliers. In [5], Galtbayar, Jensen and Yajima improve assertion (iv) : for  $n = 3$  and  $\delta > 2$ ,  $\mathcal{H}_p(U(s+1, s))$  is finite dimensional.

2 - For general  $\delta > 0$ ,  $W^\pm(s)$  do not exist and we have to define other wave operators. In ([10], [11]), Kitada and Yajima have constructed modified wave operators  $W_{H,J}^\pm$  by solving an Hamilton-Jacobi equation.

• **The case  $E_0 \neq 0$ .**

This case was studied by Moller [12] : using the Avron-Herbst formula, it suffices to examine Hamiltonians with a constant external electric field, (Stark Hamiltonians) : the spectral and the scattering theory for Stark Hamiltonians are well established [2]. In particular, a Stark Hamiltonian with a potential  $V$  satisfying  $(A_2)$  has no eigenvalues [2]. The following theorem, due to Moller, is a time-periodic version of these results.

**Theorem 2**

Assume that hypotheses  $(A_1)$ ,  $(A_2)$  are satisfied with  $\delta > \frac{1}{2}$  and with  $E_0 \neq 0$ .

Then : (i) the Floquet operators  $U(s+1, s)$  have purely absolutely continuous spectrum.

(ii) the wave operators  $W^\pm(s)$  exist for all  $s \in \mathbb{R}$  and are unitary.

(iii)  $W^\pm(s+1) = W^\pm(s)$  and  $U(s+1, s) W^\pm(s) = W^\pm(s) U_0(s+1, s)$ .

**The inverse scattering problem.**

For  $s \in \mathbb{R}$ , we define the scattering operators  $S(s) = W^{+*}(s) W^-(s)$ . It is clear that the scattering operators  $S(s)$  are periodic with period 1.

The inverse scattering problem consists to reconstruct the perturbation  $V(s, x)$  from the scattering operators  $S(s)$ ,  $s \in [0, 1]$ .

In this paper, we prove the following result :

**Theorem 3**

Assume that  $E(t)$  satisfies  $(A_1)$  and let  $V_j$ ,  $j = 1, 2$  be potentials satisfying  $(A_2)$ . We assume that  $\delta > 1$  (if  $E_0 = 0$ ),  $\delta > \frac{1}{2}$  (if  $E_0 \neq 0$  and  $n \geq 3$ ),  $\delta > \frac{3}{4}$  (if  $E_0 \neq 0$  and  $n = 2$ ). Let  $S_j(s)$  be the corresponding scattering operators.

Then :

$$\forall s \in [0, 1], S_1(s) = S_2(s) \iff V_1 = V_2 .$$

We prove Theorem 3 by studying the high energy limit of  $[S(s), p]$ , (Enss-Weder's approach [4]). We need  $n \geq 3$  in the case  $E_0 \neq 0$  in order to use the inversion of the Radon transform [6] on the orthogonal hyperplane to  $E_0$ . See also [15] for a similar problem with a Stark Hamiltonian.

We can also remark that if we know the free propagator  $U_0(t, s)$ ,  $s, t \in \mathbb{R}$ , then by virtue of the following relation :

$$(1.5) \quad S(t) = U_0(t, s) S(s) U_0(s, t) ,$$

the potential  $V(t, x)$  is uniquely reconstructed from the scattering operator  $S(s)$  at only one initial time.

In [21], Yajima proves uniqueness for the case of time-periodic potential with the condition  $\delta > \frac{n}{2} + 1$  and with  $E(t) = 0$  by studying the scattering matrices in a high energy regime.

In [20], for a time-periodic potential that decays exponentially at infinity, Weder proves uniqueness at a fixed quasi-energy.

Note also that inverse scattering for long-range time-dependent potentials without external electric fields was studied by Weder [18] with the Enss-Weder time-dependent method, and by Ito for time-dependent electromagnetic potentials for Dirac equations [8].

## 2 Proof of Theorem 3.

### 2.1 The Avron-Herbst formula.

First, let us recall some basic definitions for time-dependent Hamiltonians. Let  $\{H(t)\}_{t \in \mathbb{R}}$  be a family of selfadjoint operators on  $L^2(\mathbb{R}^n)$  such that  $\mathcal{S}(\mathbb{R}^n) \subset D(H(t))$  for all  $t \in \mathbb{R}$ .

**Definition.**

We call *propagator* a family of unitary operators on  $L^2(\mathbb{R}^n)$ ,  $U(t, s)$ ,  $t, s \in \mathbb{R}$  such that :

- 1 -  $U(t, s)$  is a strongly continuous function of  $(t, s) \in \mathbb{R}^2$ .
- 2 -  $U(t, s) U(s, r) = U(t, r)$  for all  $t, s, r \in \mathbb{R}$ .
- 3 -  $U(t, s) (\mathcal{S}(\mathbb{R}^n)) \subset \mathcal{S}(\mathbb{R}^n)$  for all  $t, s \in \mathbb{R}$ .
- 4 - If  $\Phi \in \mathcal{S}(\mathbb{R}^n)$ ,  $U(t, s)\Phi$  is continuously differentiable in  $t$  and  $s$  and satisfies :

$$i \frac{\partial}{\partial t} U(t, s) \Phi = H(t) U(t, s) \Phi , \quad i \frac{\partial}{\partial s} U(t, s) \Phi = -U(t, s) H(s) \Phi .$$

To prove the existence and the uniqueness of the propagator for our Hamiltonians  $H(t)$ , we use a generalization of the Avron-Herbst formula close to the one given in [3].

In [12], the author gives, for  $E_0 \neq 0$ , a different formula which has the advantage to be time-periodic. To study our inverse scattering problem, we use here a different one, which is defined for all  $E_0$ . We emphasize that with our choice,  $c(t)$  (see below for the definition of  $c(t)$ ) is also periodic with period 1; in particular  $c(t) = O(1)$ .

The basic idea is to generalize the well-known Avron-Herbst formula for a Stark Hamiltonian with a constant electric field  $E_0$ , [2]; if we consider the Hamiltonian  $B_0$  on  $L^2(\mathbb{R}^n)$ ,

$$(2.1) \quad B_0 = \frac{1}{2}p^2 - E_0 \cdot x ,$$

we have the following formula :

$$(2.2) \quad e^{-itB_0} = e^{-i\frac{E_0^2}{6}t^3} e^{itE_0 \cdot x} e^{-i\frac{t^2}{2}E_0 \cdot p} e^{-it\frac{p^2}{2}} .$$

In the next definition, we give a similar formula for time-dependent electric fields.

**Definition.**

We consider the family of unitary operators  $T(t)$ , for  $t \in \mathbb{R}$  :

$$T(t) = e^{-ia(t)} e^{-ib(t) \cdot x} e^{-ic(t) \cdot p} ,$$

where :

$$(2.3) \quad b(t) = - \int_0^t (E(s) - E_0) ds - \int_0^1 \int_0^t (E(s) - E_0) ds dt .$$

$$(2.4) \quad c(t) = - \int_0^t b(s) ds .$$

$$(2.5) \quad a(t) = \int_0^t \left( \frac{1}{2} b^2(s) - E_0 \cdot c(s) \right) ds .$$

**Lemma 4**

*The family  $\{H_0(t)\}_{t \in \mathbb{R}}$  has an unique propagator  $U_0(t, s)$  defined by :*

$$(2.6) \quad U_0(t, s) = T(t) e^{-i(t-s)B_0} T^*(s) .$$

**Proof.**

We can always assume  $s = 0$  and we make the following ansatz :

$$(2.7) \quad U_0(t, 0) = e^{-ia(t)} e^{-ib(t) \cdot x} e^{-ic(t) \cdot p} e^{-itB_0} .$$

Since on the Schwartz space,  $U_0(t, 0)$  must satisfy :

$$(2.8) \quad i \frac{\partial}{\partial t} U_0(t, 0) = H_0(t) U_0(t, 0) ,$$

the functions  $a(t)$ ,  $b(t)$ ,  $c(t)$  solve :

$$(2.9) \quad \dot{b}(t) = -E(t) + E_0, \quad \dot{c}(t) = -b(t), \quad \dot{a}(t) = \frac{1}{2} b^2(t) - E_0 \cdot c(t).$$

We refer to [3] for details and [12] for a different formula.  $\square$

In the same way, in order to define the propagator corresponding to the family  $\{H(t)\}$ , we consider a Stark Hamiltonian with a time-periodic potential  $V_1(t, x)$ , (we recall that  $c(t)$  is  $C^1$ -periodic function) :

$$(2.10) \quad B(t) = B_0 + V_1(t, x) \quad \text{where} \quad V_1(t, x) = e^{ic(t) \cdot p} V(t, x) e^{-ic(t) \cdot p} = V(t, x + c(t)).$$

Then,  $B(t)$  has an unique propagator  $R(t, s)$ , (see [16] for the case  $E_0 = 0$  and [12] for the case  $E_0 \neq 0$ ). It is easy to see that the propagator  $U(t, s)$  for the family  $\{H(t)\}$  is defined by :

$$(2.11) \quad U(t, s) = T(t) R(t, s) T^*(s).$$

### Comments.

Since the Hamiltonians  $H_0(t)$  and  $H(t)$  are time-periodic with period 1, one has for all  $t, s \in \mathbb{R}$  :

$$(2.12) \quad U_0(t + 1, s + 1) = U_0(t, s) \quad , \quad U(t + 1, s + 1) = U(t, s) .$$

Thus, the wave operators satisfy  $W^\pm(s + 1) = W^\pm(s)$ .

## 2.2 The high energy limit of the scattering operators.

In this section, we study the high energy limit of the scattering operators by using the well-known Enss-Weder's time-dependent method [4]. This method can be used to study Hamiltonians with electric and magnetic potentials on  $L^2(\mathbb{R}^n)$  [1], the Dirac equation [9], the N-body case [4], the Stark effect ([15], [17]), the Aharonov-Bohm effect [18].

In [13], [14] a stationary approach, based on the same ideas, is proposed to solve scattering inverse problems for Schrödinger operators with magnetic fields or with the Aharonov-Bohm effect.

Before giving the main result of this section, we need some notation.

- $\Phi, \Psi$  are the Fourier transforms of functions in  $C_0^\infty(\mathbb{R}^n)$ .
- $\omega \in S^{n-1} \cap \Pi_{E_0}$  is fixed, where  $\Pi_{E_0}$  is the orthogonal hyperplane to  $E_0$ .
- $\Phi_{\lambda, \omega} = e^{i\sqrt{\lambda}x \cdot \omega} \Phi$ ,  $\Psi_{\lambda, \omega} = e^{i\sqrt{\lambda}x \cdot \omega} \Psi$ .

We have the following high energy asymptotics where  $\langle \cdot, \cdot \rangle$  is the usual scalar product in  $L^2(\mathbb{R}^n)$  :

**Proposition 5**

*Under the assumptions of Theorem 3, we have for all  $s \in [0, 1]$ ,*

$$\langle [S(s), p] \Phi_{\lambda, \omega}, \Psi_{\lambda, \omega} \rangle = \lambda^{-\frac{1}{2}} \langle \left( \int_{-\infty}^{+\infty} \partial_x V(s, x + t\omega) dt \right) \Phi, \Psi \rangle + o(\lambda^{-\frac{1}{2}}).$$

**Comments.**

Actually, for the case  $n = 2$ ,  $E_0 \neq 0$  and  $\delta > \frac{3}{4}$ , Proposition 5 is also valid for  $\omega \in S^{n-1}$  satisfying  $|\omega \cdot E_0| < |E_0|$ , (see ([18], [15])).

Then, Theorem 3 follows from Proposition 5 and the inversion of Radon transform ([6] and [15], Section 2.3).

**Proof of Proposition 5.**

For example, let us show Proposition 5 for the case  $E_0 \neq 0$  and  $n \geq 3$ , the other cases are similar. More precisely, see [18] for the case  $E_0 = 0$ , and for the case  $n = 2$ ,  $E_0 \neq 0$ , see ([17], Theorem 2.4) and ([15], Theorem 4).

**Step 1.**

Since  $c(t)$  is periodic,  $c(t) = O(1)$ . Then,  $V_1(t, x)$  is a short-range perturbation of  $B_0$ , and we can define the usual wave operators for the pair of Hamiltonians  $(B(t), B_0)$  :

$$(2.13) \quad \Omega^\pm(s) = s - \lim_{t \rightarrow \pm\infty} R(s, t) e^{-i(t-s)B_0}.$$

Consider also the scattering operators  $S_1(s) = \Omega^{+*}(s) \Omega^-(s)$ . By virtue of (2.6) and (2.11), it is clear that :

$$(2.14) \quad S(s) = T(s) S_1(s) T^*(s).$$

Using the fact that  $e^{-ib(s) \cdot x} p e^{ib(s) \cdot x} = p + b(s)$ , we have :

$$(2.15) \quad [S(s), p] = [S(s), p + b(s)] = T(s) [S_1(s), p] T^*(s).$$

Thus,

$$(2.16) \quad \langle [S(s), p] \Phi_{\lambda, \omega}, \Psi_{\lambda, \omega} \rangle = \langle [S_1(s), p] T^*(s) \Phi_{\lambda, \omega}, T(s) \Psi_{\lambda, \omega} \rangle.$$



In other hand,

$$(2.17) \quad T^*(s) \Phi_{\lambda,\omega} = e^{i\sqrt{\lambda}x \cdot \omega} e^{ic(s) \cdot (p + \sqrt{\lambda}\omega)} e^{ib(s) \cdot x} e^{ia(s)} \Phi.$$

So, we obtain :

$$(2.18) \quad \langle [S(s), p] \Phi_{\lambda,\omega}, \Psi_{\lambda,\omega} \rangle = \langle [S_1(s), p] f_{\lambda,\omega}, g_{\lambda,\omega} \rangle,$$

where

$$(2.19) \quad f = e^{ic(s) \cdot p} e^{ib(s) \cdot x} \Phi \text{ and } g = e^{ic(s) \cdot p} e^{ib(s) \cdot x} \Psi.$$

Clearly,  $f, g$  are the Fourier transforms of functions in  $C_0^\infty(\mathbb{R}^n)$ .

• **Step 2 : Modified wave operators.**

Now, we follow a strategy close to [15] for time-dependent potentials. First, let us define a free-modified dynamic  $U_D(t, s)$  by :

$$(2.20) \quad U_D(t, s) = e^{-i(t-s)B_0} e^{-i \int_0^{t-s} V_1(u+s, up' + \frac{1}{2}u^2 E_0) du},$$

where  $p'$  is the projection of  $p$  on the orthogonal hyperplane to  $E_0$ .

We define the modified wave operators :

$$(2.21) \quad \Omega_D^\pm(s) = s - \lim_{t \rightarrow \pm\infty} R(s, t) U_D(t, s).$$

It is clear that :

$$(2.22) \quad \Omega_D^\pm(s) = \Omega^\pm(s) e^{-ig^\pm(s, p')},$$

where

$$(2.23) \quad g^\pm(s, p') = \int_0^{\pm\infty} V_1(u + s, up' + \frac{1}{2}u^2 E_0) du.$$

Thus, if we set  $S_D(s) = \Omega_D^{+*}(s) \Omega_D^-(s)$ , one has :

$$(2.24) \quad S_1(s) = e^{-ig^+(s, p')} S_D(s) e^{ig^-(s, p')}$$

• **Step 3 : High energy estimates.**

Denote  $\rho = \min(1, \delta)$ . We have the following estimations, (the proof is exactly the same as in ([15], Lemma 3) for time-independent potentials).

**Lemma 6**

For  $\lambda \gg 1$ , we have :

$$(i) \quad \left\| \left( V_1(t, x) - V_1(t, (t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right) U_D(t, s) e^{ig^\pm(s, p')} f_{\lambda,\omega} \right\| \leq C (1 + |(t-s)\sqrt{\lambda}|)^{-\frac{1}{2}-\rho}.$$

$$(ii) \quad \left\| \left( R(t, s) \Omega_D^\pm(s) - U_D(t, s) \right) e^{ig^\pm(s, p')} f_{\lambda,\omega} \right\| = O(\lambda^{-\frac{1}{2}}), \text{ uniformly for } t, s \in \mathbb{R}.$$

• **Step 4.**

We denote  $F(s, \lambda, \omega) = \langle [S_1(s), p] f_{\lambda, \omega}, g_{\lambda, \omega} \rangle$ . Using (2.24), we have :

$$\begin{aligned}
F(s, \lambda, \omega) &= \langle [e^{-ig^+(s, p')} S_D(s) e^{ig^-(s, p')}, p] f_{\lambda, \omega}, g_{\lambda, \omega} \rangle \\
&= \langle [S_D(s), p] e^{ig^-(s, p')} f_{\lambda, \omega}, e^{ig^+(s, p')} g_{\lambda, \omega} \rangle \\
&= \langle [S_D(s) - 1, p - \sqrt{\lambda} \omega] e^{ig^-(s, p')} f_{\lambda, \omega}, e^{ig^+(s, p')} g_{\lambda, \omega} \rangle \\
&= \langle (S_D(s) - 1) e^{ig^-(s, p')} (pf)_{\lambda, \omega}, e^{ig^+(s, p')} g_{\lambda, \omega} \rangle \\
&\quad - \langle (S_D(s) - 1) e^{ig^-(s, p')} f_{\lambda, \omega}, e^{ig^+(s, p')} (pg)_{\lambda, \omega} \rangle \\
&:= F_1(s, \lambda, \omega) - F_2(s, \lambda, \omega).
\end{aligned}$$

First, let us study  $F_1(s, \lambda, \omega)$ . Writing  $S_D(s) - 1 = (\Omega_D^+(s) - \Omega_D^-(s))^* \Omega_D^-(s)$  and using

$$(2.25) \quad \Omega_D^+(s) - \Omega_D^-(s) = i \int_{-\infty}^{+\infty} R(s, t) \left( V_1(t, x) - V_1(t, (t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right) U_D(t, s) dt,$$

we obtain :

$$\begin{aligned}
(2.26) \quad S_D(s) - 1 &= -i \int_{-\infty}^{+\infty} U_D(t, s)^* \left( V_1(t, x) - V_1(t, (t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right) \\
&\quad R(t, s) \Omega_D^-(s) dt.
\end{aligned}$$

Thus,

$$\begin{aligned}
F_1(s, \lambda, \omega) &= -i \int_{-\infty}^{+\infty} \langle R(t, s) \Omega_D^-(s) e^{ig^-(s, p')} (pf)_{\lambda, \omega}, \\
&\quad \left( V_1(t, x) - V_1(t, (t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right) U_D(t, s) e^{ig^+(s, p')} g_{\lambda, \omega} \rangle dt \\
&= -i \int_{-\infty}^{+\infty} \langle U_D(t, s) e^{ig^-(s, p')} (pf)_{\lambda, \omega}, \\
&\quad \left( V_1(t, x) - V_1(t, (t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right) U_D(t, s) e^{ig^+(s, p')} g_{\lambda, \omega} \rangle dt \\
&\quad + R_1(s, \lambda, \omega),
\end{aligned}$$

where :

$$\begin{aligned}
(2.27) \quad R_1(s, \lambda, \omega) &= -i \int_{-\infty}^{+\infty} \langle (R(t, s) \Omega_D^-(s) - U_D(t, s)) e^{ig^-(s, p')} (pf)_{\lambda, \omega}, \\
&\quad \left( V_1(t, x) - V_1(t, (t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right) U_D(t, s) e^{ig^+(s, p')} g_{\lambda, \omega} \rangle dt.
\end{aligned}$$

By Lemma 6, it is clear that  $R_1(s, \lambda, \omega) = O(\lambda^{-1})$ . Thus, writing  $t = \frac{\tau}{\sqrt{\lambda}} + s$ , we obtain :

$$(2.28) \quad F_1(s, \lambda, \omega) = -\frac{i}{\sqrt{\lambda}} \int_{-\infty}^{+\infty} < U_D(\frac{\tau}{\sqrt{\lambda}} + s, s) e^{ig^-(s, p')} (pf)_{\lambda, \omega} , \\ \left( V_1(\frac{\tau}{\sqrt{\lambda}} + s, x) - V_1(\frac{\tau}{\sqrt{\lambda}} + s, \frac{\tau}{\sqrt{\lambda}} p' + \frac{\tau^2}{2\lambda} E_0) \right) \\ U_D(\frac{\tau}{\sqrt{\lambda}} + s, s) e^{ig^+(s, p')} g_{\lambda, \omega} > d\tau + O(\lambda^{-1}) .$$

Denote by  $f_1(\tau, s, \lambda, \omega)$  the integrand of the (R.H.S) of (2.28). By Lemma 6 (i),

$$(2.29) \quad |f_1(\tau, s, \lambda, \omega)| \leq C (1 + |\tau|)^{-\frac{1}{2}-\rho} .$$

So, by Lebesgue's theorem, to obtain the asymptotics of  $F_1(s, \lambda, \omega)$ , it suffices to determine  $\lim_{\lambda \rightarrow +\infty} f_1(\tau, s, \lambda, \omega)$ .

Let us denote :

$$(2.30) \quad U^\pm(t, s, p') = e^{i \int_t^{\pm\infty} V_1(u+s, up' + \frac{1}{2}u^2 E_0) du} .$$

We have :

$$(2.31) \quad f_1(\tau, s, \lambda, \omega) = < e^{-i\frac{\tau}{\sqrt{\lambda}} B_0} U^-(\frac{\tau}{\sqrt{\lambda}}, s, p') (pf)_{\lambda, \omega} , \\ \left( V_1(\frac{\tau}{\sqrt{\lambda}} + s, x) - V_1(\frac{\tau}{\sqrt{\lambda}} + s, \frac{\tau}{\sqrt{\lambda}} p' + \frac{\tau^2}{2\lambda} E_0) \right) e^{-i\frac{\tau}{\sqrt{\lambda}} B_0} U^+(\frac{\tau}{\sqrt{\lambda}}, s, p') g_{\lambda, \omega} > .$$

Using the Avron-Herbst formula (2.2), we deduce that :

$$(2.32) \quad f_1(\tau, s, \lambda, \omega) = < e^{-i\frac{\tau}{2\sqrt{\lambda}} p^2} U^-(\frac{\tau}{\sqrt{\lambda}}, s, p') (pf)_{\lambda, \omega} , \\ \left( V_1(\frac{\tau}{\sqrt{\lambda}} + s, x + \frac{\tau^2}{2\lambda} E_0) - V_1(\frac{\tau}{\sqrt{\lambda}} + s, \frac{\tau}{\sqrt{\lambda}} p' + \frac{\tau^2}{2\lambda} E_0) \right) e^{-i\frac{\tau}{2\sqrt{\lambda}} p^2} U^+(\frac{\tau}{\sqrt{\lambda}}, s, p') g_{\lambda, \omega} > .$$

Then, we obtain :

$$(2.33) \quad f_1(\tau, s, \lambda, \omega) = < e^{-i\frac{\tau}{2\sqrt{\lambda}} (p + \sqrt{\lambda}\omega)^2} U^-(\frac{\tau}{\sqrt{\lambda}}, s, p' + \sqrt{\lambda}\omega) pf , \\ \left( V_1(\frac{\tau}{\sqrt{\lambda}} + s, x + \frac{\tau^2}{2\lambda} E_0) - V_1(\frac{\tau}{\sqrt{\lambda}} + s, \frac{\tau}{\sqrt{\lambda}} (p' + \sqrt{\lambda}\omega) + \frac{\tau^2}{2\lambda} E_0) \right) \\ e^{-i\frac{\tau}{2\sqrt{\lambda}} (p + \sqrt{\lambda}\omega)^2} U^+(\frac{\tau}{\sqrt{\lambda}}, s, p' + \sqrt{\lambda}\omega) g > .$$

Since

$$(2.34) \quad e^{-i\frac{\tau}{2\sqrt{\lambda}} (p + \sqrt{\lambda}\omega)^2} = e^{-i\frac{\tau\sqrt{\lambda}}{2}} e^{-i\tau\omega.p} e^{-i\frac{\tau}{2\sqrt{\lambda}} p^2} ,$$

we have

$$(2.35) \quad f_1(\tau, s, \lambda, \omega) = \langle e^{-i\frac{\tau}{2\sqrt{\lambda}}p^2} U^-(\frac{\tau}{\sqrt{\lambda}}, s, p' + \sqrt{\lambda}\omega) pf, \\ \left( V_1(\frac{\tau}{\sqrt{\lambda}} + s, x + \tau\omega + \frac{\tau^2}{2\lambda} E_0) - V_1(\frac{\tau}{\sqrt{\lambda}} + s, \frac{\tau}{\sqrt{\lambda}}(p' + \sqrt{\lambda}\omega) + \frac{\tau^2}{2\lambda} E_0) \right) \\ e^{-i\frac{\tau}{2\sqrt{\lambda}}p^2} U^+(\frac{\tau}{\sqrt{\lambda}}, s, p' + \sqrt{\lambda}\omega) g \rangle .$$

Since  $|V_1(u + s, u(p' + \sqrt{\lambda}\omega) + \frac{1}{2}u^2 E_0)| \leq C(u^2 + 1)^{-\delta} \in L^1(\mathbb{R}^+, du)$ , it is easy to show (using Lebesgue's theorem again) that :

$$(2.38) \quad s - \lim_{\lambda \rightarrow +\infty} U^\pm(\frac{\tau}{\sqrt{\lambda}}, s, p' + \sqrt{\lambda}\omega) = 1 .$$

Then,

$$(2.39) \quad \lim_{\lambda \rightarrow +\infty} f_1(\tau, s, \lambda, \omega) = \langle pf, (V_1(s, x + \tau\omega) - V_1(s, \tau\omega)) g \rangle .$$

So, we have obtained :

$$(2.40) \quad F_1(s, \lambda, \omega) = -\frac{i}{\sqrt{\lambda}} \langle pf, \left( \int_{-\infty}^{+\infty} (V_1(s, x + \tau\omega) - V_1(s, \tau\omega)) d\tau \right) g \rangle + o(\frac{1}{\sqrt{\lambda}}) .$$

In the same way, we obtain

$$(2.41) \quad F_2(s, \lambda, \omega) = -\frac{i}{\sqrt{\lambda}} \langle f, \left( \int_{-\infty}^{+\infty} (V_1(s, x + \tau\omega) - V_1(s, \tau\omega)) d\tau \right) pg \rangle + o(\frac{1}{\sqrt{\lambda}}) ,$$

so

$$(2.42) \quad F(s, \lambda, \omega) = F_1(s, \lambda, \omega) - F_2(s, \lambda, \omega)$$

$$(2.43) \quad = \frac{1}{\sqrt{\lambda}} \langle f, \left( \int_{-\infty}^{+\infty} \partial_x V_1(s, x + \tau\omega) d\tau \right) g \rangle + o(\frac{1}{\sqrt{\lambda}}) .$$

Using (2.19) and  $\partial_x V(s, x + \tau\omega) = e^{-ic(s)p} \partial_x V_1(s, x + \tau\omega) e^{ic(s)p}$ , we obtain :

$$(2.44) \quad F(s, \lambda, \omega) = \frac{1}{\sqrt{\lambda}} \langle \Phi, \left( \int_{-\infty}^{+\infty} \partial_x V(s, x + \tau\omega) d\tau \right) \Psi \rangle + o(\frac{1}{\sqrt{\lambda}}) . \square$$

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